

The helicity and vorticity of liquid crystal flows

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Abstract

We present explicit expressions of the helicity conservation in nematic liquid crystal flows, for both the Ericksen-Leslie and Landau-de Gennes theories. This is done by using a minimal coupling argument that leads to an Euler-like equation for a modified vorticity involving both velocity and structure fields (e.g. director and alignment tensor). This equation for the modified vorticity shares many relevant properties with ideal fluid dynamics and it allows for vortex filament configurations as well as point vortices in 2D. We extend all these results to particles of arbitrary shape by considering systems with fully broken rotational symmetry.

1 Introduction

Several studies on nematic liquid crystal flows have shown high velocity gradients and led to the conclusion that the coupling between the velocity $\mathbf{u}(\mathbf{x}, t)$ and structure fields is a fundamental feature of liquid crystal dynamics [27]. This conclusion has been reached from different viewpoints and by using different theories, such as the celebrated Ericksen-Leslie (EL) and the Landau-de Gennes (LdG) theories [4, 32, 31, 2]. Evidence of high velocity gradients also emerged [24] by using kinetic approaches based on the Doi model [10]. The essential difference between EL and LdG theories resides in the choice of the order parameter: while EL theory for rod-like molecules considers the dynamics of the director field $\mathbf{n}(\mathbf{x}, t)$ and it is successful in the description of low

molar-mass nematics, the LdG theory generalizes to variable molecule shapes by considering a traceless symmetric tensor field $\mathbf{Q}(\mathbf{x}, t)$. In the presence of high disclination densities the LdG theory is more reliable, since molecules may easily undergo phase transitions (e.g. from uniaxial to biaxial order) that are naturally incorporated in the theory. However, the dynamical LdG theory is not completely established and different versions are available in the literature [1, 28, 29]. Here, we shall adopt the formulation of Qian and Sheng [29], which will be simply referred to as LdG theory.

This paper considers both EL and LdG theories and it shows how the strong interplay between velocity and order parameter field reflects naturally in the helicity conservation for nematics. In this paper, the term “helicity” stands for the hydrodynamic helicity and *not* the helicity of the single liquid crystal molecule. The helicity conservation for incompressible liquid crystal flows arises from the simple velocity transformation $\mathbf{u} \rightarrow \mathbf{C} = \mathbf{u} + \mathbf{J}$, where the vector \mathbf{J} depends only on the order parameter field. The covariant vector \mathbf{C} is the total circulation (momentum per unit mass of fluid) and \mathbf{J} is the circulation associated with entrainment of fluid due to its local interaction with the nematic order parameter field. We shall show how this change of velocity variable takes the equation for the ordinary vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ into an Euler-like equation for the modified vorticity $\overline{\boldsymbol{\omega}} = \nabla \times \mathbf{C}$, thereby extending many properties of ordinary ideal fluids to nematic liquid crystals. The helicity is then given by

$$\mathcal{H} = \int (\overline{\boldsymbol{\omega}} \cdot \mathbf{C}) \, d^3\mathbf{x},$$

and this quantity naturally extends the usual expression $\int (\boldsymbol{\omega} \cdot \mathbf{u}) \, d^3\mathbf{x}$ for the helicity of three dimensional ideal flows. We recall that in Hamiltonian fluid dynamics the conservation of the hydrodynamic helicity is strictly associated to the Hamiltonian structure of the equations and holds for any Hamiltonian. This point will be further developed in the last part of the paper, where all the results will be derived directly from the Hamiltonian structure of the liquid crystal equations, see [7]. Invariant functions like the helicity are called Casimir and are of fundamental importance for the study of nonlinear stability. For two dimensional flows, such invariant quantities are given by

$$\int \Phi(\overline{\boldsymbol{\omega}}) \, d^2\mathbf{x}, \tag{1}$$

where Φ is an arbitrary smooth function. As we shall see, the same circulation concept leading to hydrodynamic helicity applies quite generally in complex

fluid theory and is related to an analogy between complex fluids and non-Abelian Yang-Mills fluid plasmas [8, 17].

In addition to helicity conservation, we present the existence of vortex-like configurations for the modified vorticity $\overline{\omega}$. Vortex structures are well known to arise in superfluid flows and their behavior is often reminiscent of disclination lines in liquid crystals. However, here we shall consider vortices that are characterized by a combination of velocity and structures fields. After extending these results to fluids with molecules of arbitrary shapes, the end of this paper discusses the geometric basis of the present treatment.

2 Director formulation

In the context of EL theory, disclinations are singularities of the director field and thus their dynamics is related to the evolution of the gradient $\nabla \mathbf{n}$. This relation has been encoded by Eringen [11] in the *wryness tensor*

$$\gamma_{\text{EL}} = \mathbf{n} \times \nabla \mathbf{n}, \quad \text{or} \quad (\gamma_{\text{EL}})_i^a = \varepsilon^{abc} n_b \times \partial_i n_c, \quad (2)$$

which identifies the amount by which the director field rotates under an infinitesimal displacement $d\mathbf{x}$. Thus, the EL wryness tensor γ_{EL} determines the spatial rotational strain [14]. In this paper we shall investigate the role of the EL wryness tensor in helicity conservation and vorticity dynamics in the EL theory. For this purpose, we ignore dissipation and concentrate on nonlinearity. This simplifies the resulting formulas. We also restrict to incompressible flows to ignore ordinary fluid thermodynamics.

Upon denoting by J the microinertia constant [11], we introduce the angular momentum variable

$$\boldsymbol{\sigma}_{\text{EL}} = J \mathbf{n} \times D_t \mathbf{n} \quad (3)$$

that is associated to the director precession. While $D_t \mathbf{n} = \partial_t \mathbf{n} + \mathbf{u} \cdot \nabla \mathbf{n}$ denotes material time-derivative, the spatial derivatives of the director field will be denoted equivalently by $\partial \mathbf{n} / \partial x^i = \partial_i \mathbf{n} = \mathbf{n}_{,i}$ depending on convenience. Upon using Einstein's summation convention, one can express the Ericksen-Leslie equations as [14, 6, 7]

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\partial_i \left(\nabla \mathbf{n}^T \cdot \frac{\partial F}{\partial \mathbf{n}_{,i}} \right) - \nabla p, \quad \nabla \cdot \mathbf{u} = 0 \quad (4)$$

$$\partial_t \boldsymbol{\sigma}_{\text{EL}} + \mathbf{u} \cdot \nabla \boldsymbol{\sigma}_{\text{EL}} = \mathbf{h} \times \mathbf{n}, \quad \partial_t \mathbf{n} + \mathbf{u} \cdot \nabla \mathbf{n} = J^{-1} \boldsymbol{\sigma}_{\text{EL}} \times \mathbf{n} \quad (5)$$

where p is the hydrodynamic pressure by which incompressibility is imposed and \mathbf{h} is the thermodynamics derivative representing the First Law response in energy to changes in the director field

$$\mathbf{h} := \frac{\partial F}{\partial \mathbf{n}} - \partial_i \left(\frac{\partial F}{\partial \mathbf{n}_{,i}} \right) .$$

The quantity F is taken to be the Oseen-Zöcher-Frank free energy

$$F = K_1(\text{div } \mathbf{n})^2 + K_2(\mathbf{n} \cdot \nabla \times \mathbf{n})^2 + K_3|\mathbf{n} \times \nabla \times \mathbf{n}|^2 . \quad (6)$$

Of course, this choice is not a limitation, because our considerations apply to a generic form of F . For example, effects of external electric and magnetic fields may be taken into account with easy modifications. The Ericksen-Leslie fluid equations follow immediately from equations (4) and (5), as shown in [6].

In Ericksen-Leslie nematodynamics, the quantity $\boldsymbol{\sigma}_{\text{EL}} \cdot \gamma_{\text{EL}}$ denotes the vector of momentum per unit mass, with components $(\boldsymbol{\sigma}_{\text{EL}} \cdot \gamma_{\text{EL}})_i = (\sigma_{\text{EL}})_a (\gamma_{\text{EL}})_i^a$. We consider the vector \mathcal{C}_{EL} defined as the sum

$$\mathcal{C}_{\text{EL}} := \mathbf{u} + \boldsymbol{\sigma}_{\text{EL}} \cdot \gamma_{\text{EL}} = \mathbf{u} + \boldsymbol{\sigma}_{\text{EL}} \cdot \mathbf{n} \times \nabla \mathbf{n} , \quad (7)$$

reminiscent of the minimal coupling formula in electromagnetic gauge theory. We observe the following equation of motion [7] (see appendix A.1): ¹

$$\partial_t \mathcal{C}_{\text{EL}} - \mathbf{u} \times \nabla \times \mathcal{C}_{\text{EL}} = -\nabla(\phi + \mathbf{u} \cdot \mathcal{C}_{\text{EL}}) , \quad (8)$$

where

$$\phi = p + F - \frac{1}{2} |\mathbf{u}|^2 - \frac{1}{2J} |\boldsymbol{\sigma}_{\text{EL}}|^2 . \quad (9)$$

¹ The same idea has been applied in superfluid plasmas, that is, in superfluid solutions whose charged condensates are coupled electromagnetically [19]. However, the similarity with gauge theory does not end with the electromagnetic analogy. The equation corresponding to (8) also follows by inspection for a Yang-Mills fluid plasma (chromohydrodynamics, cf. [8]) either from equation (2.35) or (2.49) of [17]. By this observation, chromohydrodynamics acquires a circulation theorem and the theory of complex fluids inherits an analogy with Yang-Mills fluid plasma, first noticed in [14]. This minimal coupling argument requires the wryness tensor γ_{EL} to be a connection one-form: although this is not the case for the expression $\mathbf{n} \times \nabla \mathbf{n}$, a connection one-form can be obtained by the addition of terms parallel to \mathbf{n} . By good fortune, these extra terms make no contribution in (7) because $\boldsymbol{\sigma}_{\text{EL}} \cdot \mathbf{n} = 0$.

At this point, taking the curl of equation (8) yields the Euler-like equation

$$\partial_t \bar{\omega}_{\text{EL}} + \nabla \times (\mathbf{u} \times \bar{\omega}_{\text{EL}}) = 0 \quad (10)$$

for the modified vorticity

$$\bar{\omega}_{\text{EL}} := \nabla \times \mathbf{C}_{\text{EL}} = \boldsymbol{\omega} + \nabla \times (\boldsymbol{\sigma}_{\text{EL}} \cdot \boldsymbol{\gamma}_{\text{EL}}) . \quad (11)$$

Notice that the velocity \mathbf{u} can be expressed as

$$\mathbf{u} = -\nabla \times \boldsymbol{\psi} = -\nabla \times \Delta^{-1} \bar{\omega}_{\text{EL}} + \boldsymbol{\sigma}_{\text{EL}} \cdot \boldsymbol{\gamma}_{\text{EL}} + \nabla \varphi , \quad (12)$$

where $\boldsymbol{\psi} = \Delta^{-1} \boldsymbol{\omega}$ denotes the velocity potential, which is given by the convolution of the vorticity $\boldsymbol{\omega}$ with the Green's function of the Laplace operator (analogously for $\Delta^{-1} \bar{\omega}_{\text{EL}}$). Here the pressure-like quantity φ is a scalar function arising from the term $\nabla \times \nabla \times \Delta^{-1} (\boldsymbol{\sigma}_{\text{EL}} \cdot \boldsymbol{\gamma}_{\text{EL}}) = \boldsymbol{\sigma}_{\text{EL}} \cdot \boldsymbol{\gamma}_{\text{EL}} + \nabla \varphi$ and whose only role is to keep the velocity \mathbf{u} divergence free, so that $\nabla \cdot (\boldsymbol{\sigma}_{\text{EL}} \cdot \boldsymbol{\gamma}_{\text{EL}}) = -\Delta \varphi$. The relation (12) can be inserted into equations (5) so to express the EL equations in terms of the modified vorticity $\bar{\omega}_{\text{EL}}$. An explicit expression of the quantity $\boldsymbol{\sigma}_{\text{EL}} \cdot \boldsymbol{\gamma}_{\text{EL}}$ arises from the definitions (2) and (3): $\boldsymbol{\sigma}_{\text{EL}} \cdot \boldsymbol{\gamma}_{\text{EL}} = J \nabla \mathbf{n} \cdot D_t \mathbf{n}$.

At this point we recognize that equation (10) possesses all the usual properties of Euler's equation. For example, Ertel's commuting relation

$$[D_t, \bar{\omega}_{\text{EL}} \cdot \nabla] \alpha = D_t (\bar{\omega} \cdot \nabla \alpha) - \bar{\omega} \cdot \nabla (D_t \alpha) = 0 \quad (13)$$

follows easily by direct verification, for any scalar function $\alpha(\mathbf{x}, t)$. Moreover, one has the following Kelvin circulation theorem [7]

$$\frac{d}{dt} \oint_{\Gamma(t)} (\mathbf{u} + \boldsymbol{\sigma}_{\text{EL}} \cdot \boldsymbol{\gamma}_{\text{EL}}) \cdot d\mathbf{x} = 0 , \quad (14)$$

where the line integral is calculated on a loop $\Gamma(t)$ moving with velocity \mathbf{u} . Also, conservation of the helicity [7]

$$\mathcal{H}_{\text{EL}} = \int \bar{\omega}_{\text{EL}} \cdot \mathbf{C}_{\text{EL}} \, d^3 \mathbf{x} = \int (\mathbf{u} + \boldsymbol{\sigma}_{\text{EL}} \cdot \boldsymbol{\gamma}_{\text{EL}}) \cdot \nabla \times (\mathbf{u} + \boldsymbol{\sigma}_{\text{EL}} \cdot \boldsymbol{\gamma}_{\text{EL}}) \, d^3 \mathbf{x}$$

follows from the relation

$$\partial_t (\mathbf{C}_{\text{EL}} \cdot \bar{\omega}_{\text{EL}}) + \nabla \cdot ((\mathbf{C}_{\text{EL}} \cdot \bar{\omega}_{\text{EL}}) \mathbf{u}) = -\bar{\omega}_{\text{EL}} \cdot \nabla \phi , \quad (15)$$

which is obtained by using equations (10) and (8). Integrating equation (15) over the fluid volume yields

$$\frac{d}{dt}\mathcal{H}_{\text{EL}} = - \oint\!\!\!\oint_S \phi \, \overline{\boldsymbol{\omega}}_{\text{EL}} \cdot d\mathbf{S} - \oint\!\!\!\oint_S (\mathbf{c}_{\text{EL}} \cdot \overline{\boldsymbol{\omega}}_{\text{EL}}) \mathbf{u} \cdot d\mathbf{S} \quad (16)$$

where S is the surface determined by the fluid boundary. Consequently, the right hand side of equation (16) vanishes when $\overline{\boldsymbol{\omega}}_{\text{EL}}$ and \mathbf{u} are both tangent to the boundary, thereby producing conservation of \mathcal{H}_{EL} . Remarkably, one can show that helicity conservation persists for *any* free energy $F(\mathbf{n}, \nabla \mathbf{n})$, that is the helicity \mathcal{H}_{EL} is a Casimir for EL dynamics, see [7]. At this point, a question about boundary conditions arises: while the condition of velocity tangent to the boundary is the usual condition in hydrodynamics, the condition

$$\nabla \times (\mathbf{u} + \boldsymbol{\sigma}_{\text{EL}} \cdot \mathbf{n} \times \nabla \mathbf{n}) \cdot d\mathbf{S} = 0 \quad (17)$$

emerges here for the first time. Upon denoting $\boldsymbol{\pi} = \boldsymbol{\sigma}_{\text{EL}} \times \mathbf{n}$, one has $\nabla \times (\mathbf{u} + \boldsymbol{\sigma}_{\text{EL}} \cdot \mathbf{n} \times \nabla \mathbf{n}) = \boldsymbol{\omega} + \nabla \pi_a \times \nabla n_a$, so that the boundary condition (17) reads as $\nabla \pi_a \times \nabla n_a \cdot d\mathbf{S} = -\boldsymbol{\omega} \cdot d\mathbf{S}$. This relation evidently differs from the usual “anchoring” boundary conditions (see e.g. [30]) that are widely used in the literature and for which the director alignment at the surface is insensitive to the flow. Indeed, the physical relevance of the boundary condition (17) resides in the fact that it involves *both* fluid and field variables, contrarily to other commonly available boundary conditions. The complete physical justification of (17), however, requires more study in the future.

One of the most relevant consequences of equation (10) is the existence of singular vortex-like configurations in Ericksen-Leslie nematodynamics. In two dimensions, equation (10) has the usual point vortex solution

$$\overline{\boldsymbol{\omega}}_{\text{EL}}(x, y, t) = \sum_{i=1}^N w_i \delta(x - X_i(t)) \delta(y - Y_i(t)),$$

where (X_i, Y_i) are canonically conjugate variables with respect to the Hamiltonian $\psi = \sum (\Delta^{-1} \omega)(X_i, Y_i)$. Here ψ is the potential of the velocity \mathbf{u} , satisfying EL equation (4). Upon using relation (11) and suppressing the EL

label for convenience, one expresses the Hamiltonian as

$$\begin{aligned} \psi(X_i, Y_i, \boldsymbol{\sigma}, \mathbf{n}) = & -\frac{1}{4\pi} \sum_h \left(\sum_k w_h w_k \log |(X_h - X_k, Y_h - Y_k)| \right. \\ & \left. + w_h \int \{\sigma_a, \gamma^a\}(x', y') \log |(x' - X_h, y' - Y_h)| dx' dy' \right), \end{aligned}$$

where $|(x, y)| = \sqrt{x^2 + y^2}$ and $\{\cdot, \cdot\}$ denotes the canonical Poisson bracket in (x, y) coordinates, arising from the 2D relation $\nabla \times (\boldsymbol{\sigma} \cdot \boldsymbol{\gamma}) = \nabla \times (\nabla \mathbf{n} \cdot \boldsymbol{\pi}) = \{\pi_a, n_a\}$, where $\boldsymbol{\pi} = \boldsymbol{\sigma} \times \mathbf{n}$.

Other vortex-like structures are also allowed by equation (10), e.g. vortex cores and patches. In three dimensions, the vortex filament

$$\bar{\boldsymbol{\omega}}_{\text{EL}}(\mathbf{x}, t) = \int \frac{\partial \mathbf{R}(s, t)}{\partial s} \delta(\mathbf{x} - \mathbf{R}(s, t)) ds$$

is also a solution of (10), with $\partial_t \mathbf{R} = \mathbf{u}(\mathbf{R}, t)$. The existence of these vortex structures (including vortex sheets) rise natural stability questions concerning possible equilibrium vortex configurations. Instead of pursuing this direction, which will be the subject of our future work, the next sections will show how all the above observations also hold in the LdG theory and for fluid molecules of arbitrary shape.

To conclude this section, we emphasize that in all the above discussion the velocity and the structure fields are strongly coupled together. Indeed, the singular vortex structures only exist for the vorticity $\bar{\boldsymbol{\omega}}_{\text{EL}} = \nabla \times \boldsymbol{\mathcal{C}}_{\text{EL}}$, while there is no way for the ordinary vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ or the ‘director vorticity’ $\nabla \times (\boldsymbol{\sigma}_{\text{EL}} \cdot \boldsymbol{\gamma}_{\text{EL}})$ to be singular. This strong interplay between the macro- and micromotion is the same that emerges in many of the experiments and simulations reviewed in [27].

3 The alignment tensor

In the preceding section, we investigated the hydrodynamics of a uniaxial nematic liquid crystal. At this point, it is natural to argue that in the presence of disclinations the molecules can change the configuration of their order parameter (e.g. from uniaxial to biaxial) and the EL equations cannot be used as a faithful model, which is rather given by the LdG theory based on the alignment tensor \mathbf{Q} . Several dynamical fluid models for the evolution

of the alignment tensor \mathbf{Q} are found in the literature [1, 28, 29]. In this paper we shall show how the ideal Qiang-Sheng (QS) model [29] for the LdG tensor order parameter also allows for helicity conservation, in analogy to EL theory.

The ideal QS model reads as

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\partial_l \left(\frac{\partial \mathcal{F}}{\partial \mathbf{Q}_{ij,l}} \nabla \mathbf{Q}_{ij} \right) - \nabla p, \quad \nabla \cdot \mathbf{u} = 0 \quad (18)$$

$$\partial_t \mathbf{Q} + \mathbf{u} \cdot \nabla \mathbf{Q} = J^{-1} \mathbf{P} \quad (19)$$

$$\partial_t \mathbf{P} + \mathbf{u} \cdot \nabla \mathbf{P} = -\frac{\partial \mathcal{F}}{\partial \mathbf{Q}} + \partial_i \frac{\partial \mathcal{F}}{\partial \mathbf{Q}_{,i}} - \lambda \mathbf{I} \quad (20)$$

where \mathbf{P} is conjugate to \mathbf{Q} and \mathbf{I} is the identity matrix, while λ is a Lagrange multiplier arising from the condition $\text{Tr } \mathbf{Q} = 0$. Here the free energy $\mathcal{F}(\mathbf{Q}, \nabla \mathbf{Q})$ contains the Landau-de Gennes free energy [9] as well as interaction terms with external fields. Notice that the molecular field $\partial \mathcal{F} / \partial \mathbf{Q} - \partial_i (\partial \mathcal{F} / \partial \mathbf{Q}_{,i})$ is always symmetric, so that \mathbf{P} is also symmetric at all times.

The circulation vector \mathbf{C}_{QS} for the above system is defined by

$$\mathbf{C}_{\text{QS}} := \mathbf{u} + \mathbf{P}_{ij} \nabla \mathbf{Q}_{ij}.$$

Indeed, a direct verification shows that the above vector satisfies equation (8), that is (cf appendix A.2)

$$\partial_t \mathbf{C}_{\text{QS}} + \nabla (\mathbf{u} \cdot \mathbf{C}_{\text{QS}}) - \mathbf{u} \times \nabla \times \mathbf{C}_{\text{QS}} = -\nabla \phi, \quad (21)$$

with

$$\phi = p + \mathcal{F} - \frac{1}{2} |\mathbf{u}|^2 - \frac{1}{2J} \mathbf{P}_{ij} \mathbf{P}_{ij}. \quad (22)$$

Thus, the Euler-like equation

$$\partial_t \bar{\omega}_{\text{QS}} + \nabla \times (\mathbf{u} \times \bar{\omega}_{\text{QS}}) = 0 \quad (23)$$

holds for the modified vorticity $\bar{\omega}_{\text{QS}} = \nabla \times \mathbf{C}_{\text{QS}}$. The circulation theorem

$$\frac{d}{dt} \oint_{\Gamma(t)} (\mathbf{u} + \mathbf{P}_{ij} \nabla \mathbf{Q}_{ij}) \cdot d\mathbf{x} = 0, \quad (24)$$

and the helicity conservation (for $\bar{\omega}_{\text{QS}}$ and \mathbf{u} both tangent to the boundary)

$$\frac{d}{dt} \int \mathbf{C}_{\text{QS}} \cdot \bar{\omega}_{\text{QS}} d^3 \mathbf{x} = 0$$

are a natural consequence of the Euler-like equation (23) for the QS model of LdG theory. Again Ertel’s commutation relation (13) for $\overline{\omega}_{\text{QS}}$ follows easily from equation (23). Moreover, vortex structures similar to those appearing in EL theory also exist in the LdG formulation.

At this point, one can ask about other LdG formulations and in particular one wonders whether the latter also exhibit vortex structures and conservation of hydrodynamic helicity. Among the LdG formulations of liquid crystal dynamics, the one by Beris and Edwards [1] is probably among the most common, although it is not known to possess helicity conservation. In particular, this theory treats the order parameter as a “conformation tensor field”, so that the symmetric matrix \mathbf{Q} is replaced by a symmetric tensor field on physical space. This deep geometric difference is probably responsible for the absence of the hydrodynamic helicity in the Beris-Edwards formulation. In this sense, the peculiarity of the QS model for the LdG tensor dynamics resides in exhibiting helicity conservation and its associated vorticity dynamics. These quantities both involve coupling between velocity and structure fields. The next section shows how this is actually a situation common to all fluid systems exhibiting rotational symmetry breaking.

4 Completely broken symmetries

The tensor order parameter \mathbf{Q} arises as usual from the broken rotational symmetry that is typical of liquid crystal materials. When this rotational symmetry is fully broken, one needs to account for the dynamics of the whole particle orientation, which is determined by an orthogonal matrix O (such that $O^{-1} = O^T$). This is a situation occurring, for example, in spin glass dynamics [12, 5, 22]. Eringen’s wryness tensor (here denoted by κ) is written in terms of O as [11]

$$\kappa_i^s = \frac{1}{2} \varepsilon^{mns} O_{mk} \partial_i O_{nk} = \frac{1}{2} \varepsilon^{mns} \partial_i O_{nk} O_{km}^{-1}, \quad (\text{sum over repeated indexes}) \quad (25)$$

(In this section we suppress labels such as EL or QS, in order to better adapt to the tensor index notation.) Although orthogonal matrices are difficult to work with analytically and the use of quaternions could be preferable, the correspondence between quaternions and rotation matrices is not unique. Thus, following Eringen’s work [11], we identify molecule orientations with orthogonal matrices.

It is the purpose of this section to show how equation (10) is not peculiar of nematic liquid crystals. Rather, equations of this form are peculiar of all systems with broken rotational symmetries. In particular, equation (10) also holds in the case of complete symmetry breaking, for a vorticity variable $\bar{\omega}$ depending on the fluid velocity \mathbf{u} , on the full particle orientation O and on the angular momentum vector

$$\sigma_r = \varepsilon_{rmn} O_{mk} \Psi_{nk} = \varepsilon_{rmn} \Psi_{nk} O_{km}^{-1} \quad (26)$$

where Ψ is the variable conjugate to O . In the well-known spin glass theory of Halperin and Saslow [13], the rotation matrix is small and thus it is replaced by its infinitesimal rotation angle $\boldsymbol{\theta}$, where $\exp(\boldsymbol{\theta}) = O$. Then, $\boldsymbol{\theta}$ and $\boldsymbol{\sigma}$ become canonically conjugate variables, as shown in [12]. Here we consider the whole matrix O to account for arbitrary rotations.

In the case of fully broken symmetry, the (incompressible) equations of motion can be written for an arbitrary energy density $\mathcal{E}(\boldsymbol{\sigma}, O)$ as [22]

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\sigma_r \nabla \frac{\partial \mathcal{E}}{\partial \sigma_r} + \frac{\partial \mathcal{E}}{\partial O_{mn}} \nabla O_{mn} - \nabla p \quad (27)$$

$$\partial_t \sigma_r + \mathbf{u} \cdot \nabla \sigma_r = \varepsilon_{rji} \left(\sigma_i \frac{\partial \mathcal{E}}{\partial \sigma_j} - O_{ih} \frac{\partial \mathcal{E}}{\partial O_{jh}} \right) \quad (28)$$

$$\partial_t O_{mn} + \mathbf{u} \cdot \nabla O_{mn} = -\varepsilon_{mkj} \frac{\partial \mathcal{E}}{\partial \sigma_j} O_{kn} \quad (29)$$

Notice that the validity of the above set of equations is completely general. Indeed, the above system is derived in [22] in a general fashion, under the only hypothesis that the broken symmetry group is $SO(3)$. More general broken symmetries can be certainly treated in the same way, although this paper focuses only on rotational symmetries.

In this context, equation (7) generalizes immediately by considering the wryness tensor in (25). Thus, the new circulation vector is defined by

$$\boldsymbol{\mathcal{C}} := \mathbf{u} + \boldsymbol{\sigma} \cdot \boldsymbol{\kappa} = \mathbf{u} + \frac{1}{2} \varepsilon_{mns} \sigma_s O_{mk} \nabla O_{nk}, \quad (30)$$

where \mathbf{u} , $\boldsymbol{\sigma}$ and O satisfy equations (27), (28) and (29).

At this point it is natural to ask whether the new $\boldsymbol{\mathcal{C}}$ satisfies equation (8). Remarkably, a positive answer again arises from a direct calculation by using the ordinary properties of the Levi-Civita symbol. One obtains (see

appendix [A.3](#))

$$\partial_t \mathbf{C} + \nabla (\mathbf{u} \cdot \mathbf{C}) - \mathbf{u} \times \nabla \times \mathbf{C} = -\nabla \left(p - \frac{1}{2} |\mathbf{u}|^2 \right). \quad (31)$$

Consequently, the Euler-like equation (10) holds also in this case when the rotational symmetry is completely broken. Explicitly one writes

$$\partial_t (\nabla \times \mathbf{C}) + \nabla \times (\mathbf{u} \times \nabla \times \mathbf{C}) = 0. \quad (32)$$

In turn, equations (32) and (30) imply the circulation law

$$\frac{d}{dt} \oint_{\Gamma(t)} \left(\mathbf{u} + \frac{1}{2} \varepsilon_{mns} \sigma_s O_{mk} \nabla O_{nk} \right) \cdot d\mathbf{x} = 0, \quad (33)$$

and the helicity conservation

$$\frac{d}{dt} \int_V \mathbf{C} \cdot \overline{\boldsymbol{\omega}} \, d^3\mathbf{x} = 0,$$

with $\overline{\boldsymbol{\omega}} = \nabla \times \mathbf{C}$. Thus, the existence of vortex configurations is independent of the type of symmetry breaking characterizing the fluid. Therefore, such vortices may exist in liquid crystals independently of the choice of order parameter. However, one should also emphasize that the energy conserving assumption may fail in several situations and one would then be forced to consider viscosity effects. Moreover, polymeric liquid crystals do not seem to fit easily into the present framework; rather their description requires other liquid crystal theories such as the celebrated Doi theory [10].

5 Geometric origin of the helicity invariant

In the previous sections, the helicity and vorticity of various systems with broken symmetry have been presented. However, the explicit formulation of these results still lack some more justification that can be found in the deep geometric nature of these systems, as it was emphasized in [7]. This section aims to give a brief overview of the geometric setting of the liquid crystal equations that eventually leads to the explicit formulation of their helicity and vorticity. This will show how these quantities can be found without any of the calculation presented in the Appendix, by simply relying on geometric symmetry concepts. The reader is also addressed to [26, 14].

As our starting point, we write the total Poisson bracket for a general (incompressible) fluid system with broken symmetry, involving an order parameter space M . In this case, the dynamical variables consist of the fluid momentum $\mathbf{m}(\mathbf{x})$, the order parameter state $\mathcal{Q}(\mathbf{x}) \in M$ and its conjugate variable $\mathcal{P}(\mathbf{x})$, so that $(\mathcal{Q}(\mathbf{x}), \mathcal{P}(\mathbf{x})) \in T^*M$. For simplicity, we restrict to consider the case when M is a matrix vector space. The total Poisson bracket reads as

$$\begin{aligned} \{F, G\} = & \int \mathbf{m} \cdot \left[\frac{\delta F}{\delta \mathbf{m}}, \frac{\delta G}{\delta \mathbf{m}} \right] d^3x + \int \text{Tr} \left(\left(\frac{\delta F}{\delta \mathcal{Q}} \right)^T \frac{\delta G}{\delta \mathcal{P}} - \left(\frac{\delta F}{\delta \mathcal{P}} \right)^T \frac{\delta G}{\delta \mathcal{Q}} \right) d^3x \\ & + \left\langle \frac{\delta F}{\delta(\mathcal{Q}, \mathcal{P})}, \mathcal{L}_{\frac{\delta G}{\delta \mathbf{m}}}(\mathcal{Q}, \mathcal{P}) \right\rangle - \left\langle \frac{\delta G}{\delta(\mathcal{Q}, \mathcal{P})}, \mathcal{L}_{\frac{\delta F}{\delta \mathbf{m}}}(\mathcal{Q}, \mathcal{P}) \right\rangle, \end{aligned} \quad (34)$$

where the angle bracket denotes the pairing

$$\left\langle \frac{\delta G}{\delta(\mathcal{Q}, \mathcal{P})}, \mathcal{L}_{\frac{\delta F}{\delta \mathbf{m}}}(\mathcal{Q}, \mathcal{P}) \right\rangle = \int \text{Tr} \left(\left(\frac{\delta G}{\delta \mathcal{Q}} \right)^T \mathcal{L}_{\frac{\delta F}{\delta \mathbf{m}}} \mathcal{Q} + \left(\frac{\delta G}{\delta \mathcal{P}} \right)^T \mathcal{L}_{\frac{\delta F}{\delta \mathbf{m}}} \mathcal{P} \right). \quad (35)$$

The above bracket is derived from the relabeling symmetry that characterizes all fluid systems. In particular, this bracket characterizes all Hamiltonian fluid systems with broken symmetry. The relabeling symmetry carried by the fluid emerges mathematically as an invariance property of the Hamiltonian functional $\mathcal{H} : T^*\text{Diff}(\mathbb{R}^3) \times T^*C^\infty(\mathbb{R}^3, M) \rightarrow \mathbb{R}$ under the diffeomorphism group $\text{Diff}(\mathbb{R}^3)$ of smooth invertible maps. Here the notation $C^\infty(\mathbb{R}^3, M)$ stands for the space of M -valued scalar functions, i.e. the space of order parameter fields. The reduction process induces a reduced Hamiltonian $H : \mathfrak{X}^*(\mathbb{R}^3) \times T^*C^\infty(\mathbb{R}^3, M) \rightarrow \mathbb{R}$, where $\mathfrak{X}^*(\mathbb{R}^3)$ denotes the space of differential one-forms, i.e. the space of fluid momentum vectors $\mathbf{m}(\mathbf{x})$. This process leading to the reduced Hamiltonian $H = H(\mathbf{m}, \mathcal{Q}, \mathcal{P})$ is widely explained in [26, 14, 23].

Each term in the above Poisson bracket possesses a precise geometric meaning. While the first term coincides with the Poisson bracket for ordinary fluids, the second term is the canonical bracket for the order parameter field $\mathcal{Q}(\mathbf{x})$ and its conjugated momentum $\mathcal{P}(\mathbf{x})$. Moreover, the whole second line contains the two terms arising from the action of the relabeling symmetry group $\text{Diff}(\mathbb{R}^3)$ on the canonical order parameter variables $(\mathcal{Q}(\mathbf{x}), \mathcal{P}(\mathbf{x}))$. Poisson brackets of this form were applied in different contexts, from electromagnetic charged fluids [16, 15] to superfluid dynamics [18], and even to superfluid plasmas [19].

At this point, upon following the Hamiltonian version of Noether's theorem (see [26]), one can construct the total momentum

$$\mathbf{C} = \mathbf{m} + \mathbf{J}(\mathcal{Q}, \mathcal{P})$$

where $\mathbf{J}(\mathcal{Q}, \mathcal{P})$ is the (cotangent-lift) momentum map of components

$$J_i(\mathcal{Q}, \mathcal{P}) = \text{Tr}(\mathcal{P}^T \partial_i \mathcal{Q}).$$

The geometric meaning of this momentum shift by a momentum map is best explained in [23]. In Lie derivative notation, the dynamics of the circulation quantity reads as

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_{\frac{\delta H}{\delta \mathbf{m}}} \right) \mathbf{C} = -\nabla \phi \quad (36)$$

thereby yielding Noether's conservation relation

$$\frac{d}{dt} \oint_{\Gamma(t)} \mathbf{C} \cdot d\mathbf{x} = 0$$

which then arises naturally as the circulation conservation determined by the relabeling symmetry of the system. The explicit proof of circulation theorems of this kind can be found in many works in geometric fluid dynamics; see [20] for a modern reference. After recalling that for incompressible flows $\mathbf{m} = \mathbf{u}$, it is easy to recognize that replacing M by the space $\text{Sym}_0(3)$ of traceless symmetric matrices transforms the relation (36) exactly into the relation (21), which then produces the results in Section 3. Moreover, the corresponding vorticity relation for $\overline{\omega} = d\mathbf{C}$ is easily obtained by taking the exterior differential of equation (36) and recalling that this operation commutes with Lie derivative. Then, one obtains $(\partial_t + \mathcal{L}_{\mathbf{u}}) d\mathbf{C} = 0$. The form of the helicity is also easily derived from the above arguments, upon recalling an old result in [23]. In particular, if $\mathcal{H}(\mathbf{m})$ denotes ordinary Euler's helicity, then $\mathcal{H}(\mathbf{C})$ is a Casimir invariant of the Poisson bracket (34). Notice that all the above relations hold for an arbitrary manifold M other than a matrix space. This only requires using the appropriate pairing between vectors and co-vectors.

So far, we only used the cotangent-lift momentum map, which can be found for all the cases when the dynamics involve conjugate variables in a cotangent bundle T^*M . However, this does not appear to be the case for

the discussion in Section 4, where $M = SO(3)$ and $\mathcal{Q} = O$. This apparent contradiction is easily solved by noticing that

$$\text{Tr}(\mathcal{P}^T \partial_i O) = \text{Tr}((\mathcal{P} O^{-1})^T \partial_i O O^{-1}).$$

Then, upon denoting $\hat{\sigma} = \mathcal{P} O^{-1}$ and $\hat{\kappa}_i = \partial_i O O^{-1}$, the usual isomorphism between antisymmetric matrices in the Lie algebra $\mathfrak{so}(3)$ and vectors in \mathbb{R}^3 yields the term $\boldsymbol{\sigma} \cdot \boldsymbol{\kappa}$ in the circulation quantity (30). Then, upon repeating the same steps as above, the momentum map $\text{Tr}(\mathcal{P}^T \partial_i O) = \boldsymbol{\sigma} \cdot \boldsymbol{\kappa}$ returns exactly the same results as in Section 4.

The case of nematic liquid crystals treated in Section 2 can be also obtained by a direct computation, upon setting $M = S^2/\mathbb{Z}_2$, which is the director space. Upon denoting $\boldsymbol{\pi} = J D_t \mathbf{n}$ the corresponding conjugate variable, it is easy to see that $\nabla \mathbf{n} \cdot \boldsymbol{\pi} = \boldsymbol{\sigma} \cdot \boldsymbol{\gamma}_{\text{EL}}$. However, the geometric meaning of this simple step requires more basis that can be found in [7], where this last relation is justified by Lagrangian reduction.

At this point, it is clear that the above arguments ensure the results in this paper without any need for further discussion. Nevertheless, the Appendix gives explicit proofs that can be followed without previous knowledge in geometric mechanics.

6 Conclusions

This paper provided explicit expressions for the helicity conservation in liquid crystals, in both EL and LdG theories. This conservation arises from an Euler-like equation that allows for singular vortex structures in any dimension. Some of the ideal fluid properties were extended to liquid crystal flows, e.g. Ertel's commutation relation. These results were also shown to hold for molecules of arbitrary shapes, by considering fully broken rotational symmetries occurring in some spin glass dynamics. All of the results were eventually justified by geometric symmetry arguments.

The energy-Casimir method can then be applied to study nonlinear stability properties of these systems, see [21] for several examples of how this method applies to many types of fluids. This can be used, for example, to explore the coupled macro- and micro-motion of the stationary (generalized Beltrami) solutions. While the 3D stability analysis is limited by the fact that the helicity is the only Casimir invariant, the 2D case is much richer because the whole family (1) of Casimir invariants becomes available.

One more remark concerns physically observable effects. More particularly, one wonders how conservation of total circulation causes observable effects. Even more, one would like to observe these effects in a particular experiment. A simple technique that could be used to this purpose is the use of external electric fields that drive the order-parameter variables, thereby generating fluid circulation by conservation of the total circulation. Then, if one applies an external field to a trivial motionless liquid crystal, the director alignment caused by the field would result in the generation of fluid motion.

Other physical questions also arise about the nature of vortex solutions, which evidently represent much more than simply disclinations dragged around by a smooth flow. It is possible that these solutions share many analogies with superfluid vortices in $\text{He}^3\text{-A}$, whose order parameter space is again the whole group $SO(3)$. However, the nature of these singular solutions is left open for future investigations, together with their stability properties.

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A Appendix

A.1 Derivation of equations (8) and (10)

Upon using the notation $\mathcal{L}_{\mathbf{u}}$ for the Lie derivative with respect to the velocity vector field \mathbf{u} [26], we can rewrite equations (4)-(5) as

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_{\mathbf{u}}\right) \mathbf{u} = \nabla \mathbf{n} \cdot \mathbf{h} - \nabla \left(p + F - \frac{1}{2} |\mathbf{u}|^2\right) \quad (37)$$

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_{\mathbf{u}}\right) \boldsymbol{\sigma} = \mathbf{h} \times \mathbf{n}, \quad \left(\frac{\partial}{\partial t} + \mathcal{L}_{\mathbf{u}}\right) \mathbf{n} = J^{-1} \boldsymbol{\sigma} \times \mathbf{n}. \quad (38)$$

Then, one simply calculates

$$\begin{aligned}
\left(\frac{\partial}{\partial t} + \mathcal{L}_{\mathbf{u}}\right) \mathbf{C}_{\text{EL}} &= \left(\frac{\partial}{\partial t} + \mathcal{L}_{\mathbf{u}}\right) (\mathbf{u} + \boldsymbol{\sigma}_{\text{EL}} \cdot \mathbf{n} \times \nabla \mathbf{n}) \\
&= \nabla \mathbf{n} \cdot \mathbf{h} - \nabla \left(p + F - \frac{1}{2} |\mathbf{u}|^2 \right) + \mathbf{h} \times \mathbf{n} \cdot \mathbf{n} \times \nabla \mathbf{n} \\
&\quad - J^{-1} \boldsymbol{\sigma}_{\text{EL}} \cdot \nabla \mathbf{n} \times (\boldsymbol{\sigma}_{\text{EL}} \times \mathbf{n}) + J^{-1} \boldsymbol{\sigma}_{\text{EL}} \cdot \mathbf{n} \times (\nabla \boldsymbol{\sigma}_{\text{EL}} \times \mathbf{n}) \\
&\quad + J^{-1} \boldsymbol{\sigma}_{\text{EL}} \cdot \mathbf{n} \times (\boldsymbol{\sigma}_{\text{EL}} \times \nabla \mathbf{n}) .
\end{aligned}$$

At this point, standard vector identities yield

$$\begin{aligned}
\mathbf{h} \times \mathbf{n} \cdot \mathbf{n} \times \nabla \mathbf{n} &= (\mathbf{h} \cdot \mathbf{n}) (\nabla \mathbf{n} \cdot \mathbf{n}) - (\nabla \mathbf{n} \cdot \mathbf{h}) (\mathbf{n} \cdot \mathbf{n}) \\
&= -\nabla \mathbf{n} \cdot \mathbf{h} \\
\nabla \mathbf{n} \times (\boldsymbol{\sigma}_{\text{EL}} \times \mathbf{n}) &= (\nabla \mathbf{n} \cdot \mathbf{n}) \boldsymbol{\sigma}_{\text{EL}} - (\nabla \mathbf{n} \cdot \boldsymbol{\sigma}_{\text{EL}}) \mathbf{n} \\
&= -(\nabla \mathbf{n} \cdot \boldsymbol{\sigma}_{\text{EL}}) \mathbf{n} \\
\mathbf{n} \times (\nabla \boldsymbol{\sigma}_{\text{EL}} \times \mathbf{n}) &= (\mathbf{n} \cdot \mathbf{n}) \nabla \boldsymbol{\sigma}_{\text{EL}} - (\nabla \boldsymbol{\sigma}_{\text{EL}} \cdot \mathbf{n}) \mathbf{n} \\
&= \nabla \boldsymbol{\sigma}_{\text{EL}} - (\nabla \boldsymbol{\sigma}_{\text{EL}} \cdot \mathbf{n}) \mathbf{n} \\
\mathbf{n} \times (\boldsymbol{\sigma}_{\text{EL}} \times \nabla \mathbf{n}) &= (\nabla \mathbf{n} \cdot \mathbf{n}) \boldsymbol{\sigma}_{\text{EL}} - (\boldsymbol{\sigma}_{\text{EL}} \cdot \mathbf{n}) \nabla \mathbf{n} \\
&= 0 ,
\end{aligned}$$

where we have made use of the relations $|\mathbf{n}|^2 = 1$ and $\boldsymbol{\sigma}_{\text{EL}} \cdot \mathbf{n} = 0$. Therefore equation (8) follows directly from

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_{\mathbf{u}}\right) \mathbf{C}_{\text{EL}} = -\nabla \left(p + F - \frac{1}{2} |\mathbf{u}|^2 - \frac{1}{2J} |\boldsymbol{\sigma}_{\text{EL}}|^2 \right) .$$

The equation (10) follows by taking the curl of the above equation, upon recalling that the curl is given by the exterior differential, so that $\mathbf{d}(\mathbf{C}_{\text{EL}} \cdot d\mathbf{x}) = (\nabla \times \mathbf{C}_{\text{EL}}) \cdot d\mathbf{S}$. Since the differential always commutes with the Lie derivative [26], equation (10) follows immediately. It is also easy to see that equation (15) arises by calculating $(\partial_t + \mathcal{L}_{\mathbf{u}}) (\mathbf{C}_{\text{EL}} \cdot \overline{\boldsymbol{\omega}}_{\text{EL}}) = \overline{\boldsymbol{\omega}}_{\text{EL}} \cdot \nabla \phi$.

A.2 Derivation of equations (21) and (23)

Upon introducing the Lie derivative notation, the equations (18)-(19)-(20) read as

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_{\mathbf{u}}\right) \mathbf{u} = \mathbf{h}_{ij} \nabla Q_{ij} - \nabla \left(p + \mathcal{F} - \frac{1}{2} |\mathbf{u}|^2\right), \quad \nabla \cdot \mathbf{u} = 0 \quad (39)$$

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_{\mathbf{u}}\right) \mathbf{Q} = J^{-1} \mathbf{P} \quad (40)$$

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_{\mathbf{u}}\right) \mathbf{P} = -\mathbf{h} - \lambda \mathbf{I} \quad (41)$$

where we have denoted the molecular field by

$$\mathbf{h} = \frac{\partial \mathcal{F}}{\partial \mathbf{Q}} - \partial_i \frac{\partial \mathcal{F}}{\partial Q_{,i}}.$$

Then, one simply calculates

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \mathcal{L}_{\mathbf{u}}\right) \mathbf{C}_{\text{QS}} &= \left(\frac{\partial}{\partial t} + \mathcal{L}_{\mathbf{u}}\right) \left(\mathbf{u} + \mathbf{P}_{ij} \nabla Q_{ij}\right) \\ &= \mathbf{h}_{ij} \nabla Q_{ij} - \nabla \left(p + \mathcal{F} - \frac{1}{2} |\mathbf{u}|^2\right) - \mathbf{h}_{ij} \nabla Q_{ij} - \lambda \delta_{ij} \nabla Q_{ij} + \frac{1}{J} \mathbf{P}_{ij} \nabla P_{ij}, \end{aligned}$$

which becomes

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_{\mathbf{u}}\right) \mathbf{C}_{\text{QS}} = -\nabla \left(p + \mathcal{F} + \lambda \delta_{ij} Q_{ij} - \frac{1}{2} |\mathbf{u}|^2 - \frac{1}{2J} \mathbf{P}_{ij} \mathbf{P}_{ij}\right).$$

Finally, taking the curl of the above equation returns (23).

A.3 Derivation of equation (31)

Upon using the Lie derivative notation, equations (27)-(28)-(29) may be written as

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_{\mathbf{u}}\right) \mathbf{u} = -\sigma_r \nabla \frac{\partial \mathcal{E}}{\partial \sigma_r} + \frac{\partial \mathcal{E}}{\partial O_{mn}} \nabla O_{mn} - \nabla \left(p - \frac{1}{2} |\mathbf{u}|^2\right) \quad (42)$$

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_{\mathbf{u}}\right) \sigma_r = \varepsilon_{rji} \left(\sigma_i \frac{\partial \mathcal{E}}{\partial \sigma_j} - O_{ih} \frac{\partial \mathcal{E}}{\partial O_{jh}}\right) \quad (43)$$

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_{\mathbf{u}}\right) O_{mn} = -\varepsilon_{mkj} \frac{\partial \mathcal{E}}{\partial \sigma_j} O_{kn} \quad (44)$$

so that, upon denoting $\phi = p - |\mathbf{u}|^2/2$, one computes

$$\begin{aligned}
\left(\frac{\partial}{\partial t} + \mathcal{L}_{\mathbf{u}}\right) \mathbf{C} &= \left(\frac{\partial}{\partial t} + \mathcal{L}_{\mathbf{u}}\right) \left(\mathbf{u} + \frac{1}{2} \varepsilon_{mns} \sigma_s O_{mk} \nabla O_{nk}\right) \\
&= -\sigma_r \nabla \frac{\partial \mathcal{E}}{\partial \sigma_r} + \frac{\partial \mathcal{E}}{\partial O_{mn}} \nabla O_{mn} - \nabla \phi \\
&\quad + \frac{1}{2} \varepsilon_{mns} \left(\varepsilon_{sji} \sigma_i \frac{\partial \mathcal{E}}{\partial \sigma_j} - \varepsilon_{sji} O_{ih} \frac{\partial \mathcal{E}}{\partial O_{jh}} \right) O_{mk} \nabla O_{nk} \\
&\quad - \frac{1}{2} \varepsilon_{mns} \sigma_s \varepsilon_{mhj} \frac{\partial \mathcal{E}}{\partial \sigma_j} O_{hk} \nabla O_{nk} \\
&\quad - \frac{1}{2} \varepsilon_{mns} \sigma_s O_{mk} \varepsilon_{nhj} \left(\nabla \frac{\partial \mathcal{E}}{\partial \sigma_j} O_{hk} + \frac{\partial \mathcal{E}}{\partial \sigma_j} \nabla O_{hk} \right).
\end{aligned}$$

At this point we observe that, since $O_{mk} O_{hk} = O_{mk} O_{kh}^{-1} = \delta_{mh}$, then

$$\begin{aligned}
-\sigma_r \nabla \frac{\partial \mathcal{E}}{\partial \sigma_r} - \frac{1}{2} \varepsilon_{mns} \sigma_s O_{mk} \varepsilon_{nhj} \nabla \frac{\partial \mathcal{E}}{\partial \sigma_j} O_{hk} &= -\sigma_r \nabla \frac{\partial \mathcal{E}}{\partial \sigma_r} - \frac{1}{2} \varepsilon_{hns} \varepsilon_{nhj} \sigma_s \nabla \frac{\partial \mathcal{E}}{\partial \sigma_j} \\
&= -\sigma_r \nabla \frac{\partial \mathcal{E}}{\partial \sigma_r} + \delta_{sj} \sigma_s \nabla \frac{\partial \mathcal{E}}{\partial \sigma_j} = 0.
\end{aligned}$$

Moreover, the sum of all terms in $\partial \mathcal{E} / \partial \sigma$ can be written as

$$\begin{aligned}
&\sigma_s \frac{\partial \mathcal{E}}{\partial \sigma_j} (\varepsilon_{ijs} \varepsilon_{mni} O_{mk} \nabla O_{nk} - \varepsilon_{ims} \varepsilon_{inj} O_{nk} \nabla O_{mk} - \varepsilon_{mis} \varepsilon_{inj} \nabla O_{nk} O_{mk}) \\
&= -\sigma_s \frac{\partial \mathcal{E}}{\partial \sigma_j} \varepsilon_{ijs} \varepsilon_{imn} O_{mk} \nabla O_{nk} \\
&= -\sigma_s \frac{\partial \mathcal{E}}{\partial \sigma_j} (\delta_{jm} \delta_{sn} - \delta_{jn} \delta_{sm}) O_{mk} \nabla O_{nk} \\
&= -\sigma_s \frac{\partial \mathcal{E}}{\partial \sigma_j} (\nabla O_{sk} O_{kj}^{-1} + O_{sk} \nabla O_{kj}^{-1}) \\
&= -\sigma_s \frac{\partial \mathcal{E}}{\partial \sigma_j} \nabla (O_{sk} O_{kj}^{-1}) = 0.
\end{aligned}$$

In addition, we calculate

$$\begin{aligned}
& \frac{\partial \mathcal{E}}{\partial O_{mn}} \nabla O_{mn} - \frac{1}{2} \varepsilon_{sji} \varepsilon_{mns} O_{ih} \frac{\partial \mathcal{E}}{\partial O_{jh}} O_{mk} \nabla O_{nk} \\
&= \frac{\partial \mathcal{E}}{\partial O_{mn}} \nabla O_{mn} + \frac{1}{2} (\delta_{mj} \delta_{ni} - \delta_{mi} \delta_{nj}) O_{ih} \frac{\partial \mathcal{E}}{\partial O_{jh}} O_{mk} \nabla O_{nk} \\
&= \frac{\partial \mathcal{E}}{\partial O_{mn}} \nabla O_{mn} + \frac{1}{2} (O_{ih} O_{jk} \nabla O_{ik} - O_{ih} O_{ik} \nabla O_{jk}) \frac{\partial \mathcal{E}}{\partial O_{jh}} \\
&= \frac{\partial \mathcal{E}}{\partial O_{mn}} \nabla O_{mn} + \frac{1}{2} (O_{hi}^{-1} \nabla O_{ik} O_{kj}^{-1} - \delta_{hk} \nabla O_{jk}) \frac{\partial \mathcal{E}}{\partial O_{jh}} \\
&= \frac{\partial \mathcal{E}}{\partial O_{mn}} \nabla O_{mn} - \frac{1}{2} (\nabla O_{hj}^{-1} + \nabla O_{jh}) \frac{\partial \mathcal{E}}{\partial O_{jh}} \\
&= \frac{\partial \mathcal{E}}{\partial O_{mn}} \nabla O_{mn} - \nabla O_{jh} \frac{\partial \mathcal{E}}{\partial O_{jh}} = 0.
\end{aligned}$$

Thus, we have proved the relation

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_u \right) \mathbf{c} = -\nabla \phi,$$

whose curl yields the corresponding Euler-like equation (32), thereby recovering the corresponding helicity conservation. Notice that the above result also holds in the case of explicit dependence of the free energy \mathcal{E} on the gradient ∇O of the orientational order parameter.

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